Published by IOP Publishing for SISSA

Received: *August 19, 2008*
Accepted: *September 29, 2008*
Published: *October 10, 2008*

The one-loop H^2R^3 and $H^2(\nabla H)^2R$ terms in the effective action

David M. Richards

Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, U.K.
E-mail: D.M.Richards@damtp.cam.ac.uk

Abstract: We consider the one-loop B^2h^3 and B^4h amplitudes in type II string theory, where B is the $NS \otimes NS$ two-form and h the graviton, and expand to lowest order in α'. After subtracting diagrams due to quartic terms in the effective action, we determine the presence and structure of both an H^2R^3 and $H^2(\nabla H)^2R$ term. We show that both terms are multiplied by the usual $(t_8t_8 \pm \frac{1}{8}\epsilon_{10}\epsilon_{10})$ factor.

Keywords: Superstrings and Heterotic Strings, Supersymmetric Effective Theories.

JHEP10(2008)043

Contents

1. Introduction **1**

2. The effective action from four-point amplitudes **2**

3. The B^2h^3 amplitude and its low-energy expansion **4**

4. Consequences for the effective action **7**

4.1 Expansion of various tensors 8

4.2 Propagators 9

4.3 Evaluation of diagrams 10

4.4 New H^2R^3 terms 13

5. The B^4h amplitude **15**

6. The $\epsilon_8\epsilon_8$ terms **16**

JHEP10(2008)043

1. Introduction

In a recent paper [1] we considered the one-loop five-graviton amplitude, h^5, in type II string theory. From its low-energy expansion we concluded that R^5 and ∇^2R^5 terms are absent from the effective action, but that ∇^4R^5 terms are present, where R is the Riemann tensor. This paper considers similar amplitudes but with $NS \otimes NS$ two-form potentials, B, replacing some of the gravitons. In particular we consider the one-loop B^2h^3 and B^4h amplitudes and expand to lowest order in α'. This reveals the presence and tensor structure of H^2R^3 and $H^2(\nabla H)^2R$ terms in the effective action, where H is the field strength associated with B.

For our purposes, the low-energy effective action is a functional of the massless spectrum of string theory such that, even for string loops, only *tree* diagrams are required to reproduce string amplitudes. This is the sense in which the famous R^4 correction to supergravity, as in for example [2], should be interpreted. After expanding an amplitude for small α', new terms in the effective action can only be determined after diagrams due to previously-known terms are subtracted. For example, before we can find H^2R^3 from the lowest-order expansion of the B^2h^3 amplitude, it is necessary to subtract diagrams involving quartic effective action terms, such as the $(\nabla H)^2R^2$ term.

We calculate amplitudes using the light-cone gauge GS formalism, which requires that k^+ vanishes for all external states. As a consequence, there are certain terms, both in amplitudes and in the effective action, that cannot be discovered. For example, $\epsilon_{10}\epsilon_{10}$

terms with fewer than two contractions between the epsilons, such as (3.13) in [3], will be missed. Similarly, the Bh^4 amplitude, and hence the one-loop $B \wedge t_8 R^4$ term found in [4], will not be found. However, all other terms, and in particular $\epsilon_{10}\epsilon_{10}$ terms with at least two contractions between the epsilons, will be seen.

The plan of this paper is as follows. In section 2 we review the one-loop four-graviton amplitude, the associated R^4 term in the effective action, and its extension to include $NS \otimes NS$ two-forms found in [5]. The B^2h^3 amplitude is calculated in section 3 and then expanded to lowest-order in α'. Section 4 is concerned with expanding the quartic effective action and calculating the relevant diagrams. After these are subtracted, the remaining terms are covariantised to discover a new $t_8t_8H^2R^3$ term. The whole analysis is extended to the B^4h case in section 5, which results in a new $t_8t_8H^2(\nabla H)^2R$ term. Finally, section 6 pays closer attention to ϵ_8 terms in the amplitudes. This shows that the t_8t_8 in both H^2R^3 and $H^2(\nabla H)^2R$ should be generalised to $(t_8t_8 \pm \frac{1}{8}\epsilon_{10}\epsilon_{10})$, with $+/-$ for IIB/IIA, where one pair of indices is contracted between the epsilon tensors. Throughout we will often set $2\alpha' = 1$.

2. The effective action from four-point amplitudes

Before considering amplitudes for five states, we review terms in the effective action which arise from four-particle amplitudes involving gravitons and $NS \otimes NS$ two-forms. For the case of four gravitons the one-loop amplitude is well-known to be given by [6]

$$A_{4h} = \hat{K} \int \frac{d^2\tau}{(\mathrm{Im}\,\tau)^5} \int \prod_{r=1}^{3} d^2v_r \prod_{r<s} (\chi_{rs})^{\frac{1}{2}k_r \cdot k_s}, \tag{2.1}$$

with

$$\hat{K} = t_8^{a_1b_1\cdots a_4b_4} t_8^{c_1d_1\cdots c_4d_4} k^1_{a_1} k^1_{c_1} h^1_{b_1d_1} k^2_{a_2} k^2_{c_2} h^2_{b_2d_2} k^3_{a_3} k^3_{c_3} h^3_{b_3d_3} k^4_{a_4} k^4_{c_4} h^4_{b_4d_4}, \tag{2.2}$$

where the four gravitons have polarisations $h^r_{a_rb_r}$ and momenta $k^r_{a_r}$, and r ranges from 1 to 4. Here v_r are the positions of the vertex operators on the torus and their integrals are taken over the rectangular region $-\frac{1}{2} < \mathrm{Re}v \leq \frac{1}{2}, -\frac{1}{2}\mathrm{Im}\,\tau < \mathrm{Im}\,v \leq \frac{1}{2}\mathrm{Im}\,\tau$; whereas the variable τ parameterizes the modulus of the torus and so is integrated over a fundamental domain of $\mathrm{SL}(2,\mathbb{Z})$. The function $\chi_{rs} \equiv \chi(v_r - v_s, \tau)$ is a non-singular, doubly periodic function of v and $\bar{v}$ which is given explicitly in [6]. The t_8 tensor originates from a trace over eight fermionic zero modes and can be written as a sum of an ϵ_8 tensor and sixty $\delta\delta\delta\delta$ tensors [6]. Often t_8 is defined without the ϵ_8 tensor, especially when written in effective actions, and we will clarify this issue later. However, for the four-graviton amplitude this difference is not important since the ϵ parts vanish by momentum conservation.

As shown in [7−9], the integrals in (2.1) only converge for $s = t = u = 0$ where s, t, u are the usual four-particle Mandelstam variables defined in [1]. Even for complex values of the momenta, the convergence is only for purely imaginary values of s, t and u. The resolution is to analytically continue from the imaginary axis to the entire complex plane. Only then can the amplitude be shown to contain the correct massive poles and threshold cuts demanded by unitarity.

The low-energy expansion of one-loop amplitudes can be quite involved [10, 1], but since we only require the expansion at lowest-order in α' the situation is much simpler. To find the lowest-order expansion of (2.1) we set $k_r \cdot k_s$ to zero for all r, s giving

$$A_{4h}|_{\alpha'^3} = \hat{K} \int \frac{d^2\tau}{(\operatorname{Im}\tau)^2} = \frac{\pi}{3}\hat{K}, \tag{2.3}$$

where the power of α' is, as throughout this paper, relative to the Einstein-Hilbert term. It is trivial to covariantise this and find the famous $t_8 t_8 R^4$ term in the effective action. If this one-loop term is combined with the equivalent tree-level result found in [11] then the α'^3 term is given in Einstein frame by

$$\alpha'^3 \int d^{10}x\sqrt{-g}\left(2\zeta(3)e^{-3\phi/2} + \frac{2\pi^2}{3}e^{\phi/2}\right)\mathcal{R}^4, \tag{2.4}$$

where $\mathcal{R}^4$ is shorthand for

$$t_8^{a_1b_1a_2b_2a_3b_3a_4b_4} t_8^{c_1d_1c_2d_2c_3d_3c_4d_4} R_{a_1b_1c_1d_1} R_{a_2b_2c_2d_2} R_{a_3b_3c_3d_3} R_{a_4b_4c_4d_4}, \tag{2.5}$$

and where we have fixed the normalisation for the string S-matrix so that the one-loop term contains an extra factor of 2π relative to the tree-level term [12].

In the case of IIB it is possible to extend the $\mathcal{R}^4$ term to all orders in the string coupling, even including non-perturbative effects. It was shown in [2, 13–15] that the complete $\mathcal{R}^4$ action is given by $\alpha'^3 \int d^{10}x\sqrt{-g}\, Z_{3/2}(\tau,\bar{\tau})\mathcal{R}^4$, where $Z_{3/2}$ is a non-holomorphic Eisenstein series given by

$$\begin{aligned} Z_{3/2}(\tau,\bar{\tau}) &= \sum_{(m,n)\neq(0,0)} \frac{(\operatorname{Im}\tau)^{3/2}}{|m\tau+n|^3} \\ &= 2\zeta(3)e^{-3\phi/2} + \frac{2\pi^2}{3}e^{\phi/2} \\ &\quad + 4\pi\sum_{k\neq 0}\mu(k)e^{-2\pi(|k|e^{-\phi} - ikC^{(0)})}k^{1/2}\left(1 + \frac{3}{16\pi|k|}e^{\phi} + \ldots\right), \end{aligned} \tag{2.6}$$

with $\mu(k) = \sum_{d|k} d^{-2}$. Here τ, which should not be confused with the modular parameter in one-loop amplitudes, is the usual combination of the Ramond-Ramond scalar and the dilaton, $\tau = C^{(0)} + ie^{-\phi}$. The expansion shows that there are no perturbative contributions beyond one-loop, but that there are an infinite sum of single D-instanton terms, with characteristic $e^{1/g}$ behaviour, which were first studied in [2].

Four-particle amplitudes containing $NS \otimes NS$ two-forms were first studied in [5]. The result is identical to (2.1) but with the relevant replacements of h_{ab} by B_{ab} in $\hat{K}$. Amplitudes with an odd number of B fields trivially vanish since $\hat{K}$ changes sign under $(a_r, b_r) \leftrightarrow (c_r, d_r)$. [5], where terms involving dilatons were also studied, showed that the lowest-order contributions to the effective action can be written exactly as in (2.4) and (2.5) but with R_{abcd} everywhere replaced by

$$\bar{R}_{abcd} = R_{abcd} + \frac{1}{2}e^{-\phi/2}\nabla_{[a}H_{b]cd} - \frac{1}{4}g_{[a_{[c}}\nabla_{b]}\nabla_{_{d]}}\phi, \tag{2.7}$$

where $H_{abc} \equiv 3\nabla_{[a}B_{bc]}$ is the field strength associated with B_{ab}. This leads to various new terms such as $R^2(\nabla H)^2$, $(\nabla H)^4$ and $R^3\nabla\nabla\phi$. The vanishing of terms involving an odd number of H fields follows from parity.[1]

3. The B^2h^3 amplitude and its low-energy expansion

Using the light-cone gauge Green-Schwarz formalism, we now calculate similar one-loop amplitudes but with five rather than four states. The case of five gravitons was considered in [1]. Here we will replace some of these gravitons by $NS\otimes NS$ two-forms. It is no longer true that amplitudes with an odd number of B fields, such as Bh^4, will vanish. However, the non-zero piece will be entirely contained in the $\epsilon_{10}t_8$ part; the t_8t_8 and $\epsilon_{10}\epsilon_{10}$ terms will still vanish. Since, as mentioned above, the $\epsilon_{10}t_8$ pieces cannot be seen using this formalism, we instead choose to focus on the B^2h^3 and B^4h cases.

The B^2h^3 amplitude proceeds exactly as in the five-graviton case in [1], simply with h_1 and h_2 replaced by B_1 and B_2. Here we only sketch the calculation and refer the reader to [1] for details. Let the two-forms have polarisations B_1, B_2 and momenta k_1, k_2 respectively, and let the gravitons have polarisations h_1, h_2, h_3 and momenta k_3, k_4, k_5 respectively. The two-forms differ from the gravitons in that their polarisations are antisymmetric rather than symmetric. Both the graviton and $NS\otimes NS$ two-form vertex operators are given by [16]

$$\mathcal{V}_{h,B}(k,z) = \zeta_{ac}(\partial X^a(z) - R^{ab}(z)k^b)(\bar\partial X^c(z) - \tilde R^{cd}(z)k^d)e^{ik\cdot X(z)}, \tag{3.1}$$

where ζ_{ab} is the polarisation and $R^{ab}(z) \equiv \frac{1}{4}S(z)^A\gamma^{ab}_{AB}S^B(z)$. Motivated by the usual prescription for calculating GS amplitudes, explained in [6], we consider

$$A_{B^2h^3} = \int d^2\tau \int \prod_{r=1}^{4} d^2v_r \int d^{10}p\,\mathrm{Tr}\left(\mathcal{V}_B(k_1,\rho_1)\mathcal{V}_B(k_2,\rho_2)\mathcal{V}_h(k_3,\rho_3)\cdots w^{L_0}\bar w^{\tilde L_0}\right), \tag{3.2}$$

where $v_r = \ln\rho_r/2\pi i$, $\tau = \ln w/2\pi i$, and the trace is over all α, $\tilde\alpha$, S and $\tilde S$ modes. The trace over S vanishes unless there are at least eight S_0 zero modes (and similarly for $\tilde S$) and so there are only three types of term to consider: one term containing $R^5\tilde R^5$, ten terms containing $\partial XR^4\tilde R^5$ or $R^5\bar\partial X\tilde R^4$, and twenty-five terms containing $\partial XR^4\bar\partial X\tilde R^4$.

If we suppress the polarisation tensors and perform the traces and p-integral, then the $R^5\tilde R^5$ term can be evaluated as

$$\int \frac{d^2\tau}{(\mathrm{Im}\,\tau)^5}\int\prod_{r=1}^{4}d^2v_r\prod_{r<s}(\chi_{rs})^{\frac12 k_r\cdot k_s}k_1^{b_1}\cdots k_5^{b_5}k_1^{d_1}\cdots k_5^{d_5}$$
$$\times\left(t_{10}^{a_1b_1a_2b_2\cdots}+\sum_{r<s}\bar t_{10}^{a_rb_ra_sb_s\cdots}\eta'(v_{rs},\tau)\right)\left(t_{10}^{c_1d_1c_2d_2\cdots}+\sum_{r<s}\bar t_{10}^{c_rd_rc_sd_s\cdots}\bar\eta'(v_{rs},\tau)\right), \tag{3.3}$$

where, for example, $t_{10}^{a_2b_2a_4b_4\cdots}$ is shorthand for $t_{10}^{a_2b_2a_4b_4a_1b_1a_3b_3a_5b_5}$. Here t_{10} and $\bar t_{10}$ are ten-index tensors, which can both be written as sums of $t_8\delta$ tensors, as given in the appendix

[1] This only applies to t_8t_8 and $\epsilon_{10}\epsilon_{10}$ terms. Terms involving a single ϵ_{10} are not forbidden, such as the $B\wedge t_8R^4$ term found in IIA [4].

of [1]. The function $\eta'(v_{rs},\tau)$ can be expressed in terms of Jacobi theta functions via $\pi i(2\eta'(v,\tau)+1) = -\theta_1'(v,\tau)/\theta_1(v,\tau)$. Similarly, the $\partial X(\rho_1)R^4\tilde{R}^5$ term gives

$$\int \frac{d^2\tau}{(\mathrm{Im}\,\tau)^5}\int\prod_{r=1}^{4} d^2v_r \prod_{r<s}(\chi_{rs})^{\frac{1}{2}k_r\cdot k_s} k_2^{b_2}\cdots k_5^{b_5} k_1^{d_1}\cdots k_5^{d_5}$$
$$\times\, t_8^{a_2b_2\cdots a_5b_5}\left(\sum_{r\neq 1} k_r^{a_1}\eta(v_{r1},\tau)\right)\left(t_{10}^{c_1d_1c_2d_2\cdots} + \sum_{r<s}\bar{t}_{10}^{c_rd_rc_sd_s\cdots}\bar{\eta}'(v_{rs},\tau)\right), \quad (3.4)$$

where $\eta(v,\tau) = -\eta'(v,\tau) + \frac{\mathrm{Im}\,v}{\mathrm{Im}\,\tau} - \frac{1}{2}$. Other $\partial X R^4\tilde{R}^5$ terms and the $R^5\bar{\partial}X\tilde{R}^4$ terms are given by similar expressions. Finally, evaluating the traces and p-integral for, for example, $\partial X(\rho_1)R^4\bar{\partial}X(\rho_2)\tilde{R}^4$ gives

$$\int \frac{d^2\tau}{(\mathrm{Im}\,\tau)^5}\int\prod_{r=1}^{4} d^2v_r \prod_{r<s}(\chi_{rs})^{\frac{1}{2}k_r\cdot k_s} k_2^{b_2}\cdots k_5^{b_5} k_1^{d_1} k_3^{d_3}\cdots k_5^{d_5} \quad (3.5)$$
$$\times\left(\sum_{r=2}^{5} k_r^{a_1}\eta(v_{r1},\tau)\sum_{s=1,s\neq 2}^{5} k_s^{c_2}\bar{\eta}(v_{s2},\tau) - 2\delta^{a_1c_2}\hat{\Omega}(v_{12},\tau)\right) t_8^{a_2b_2a_3b_3\cdots} t_8^{c_1d_1c_3d_3\cdots},$$

where $\hat{\Omega}(v,\tau) = -1/(2\pi\mathrm{Im}\,\tau)$. Again there are similar expressions for the other $\partial X R^4\bar{\partial}X\tilde{R}^4$ terms.

Using various identities given in the appendix of [1], both t_{10} and $\bar{t}_{10}$ can be eliminated in favour of t_8 tensors. This allows the full amplitude to be packaged together as

$$A_{B^2h^3,t_8t_8} = B^1_{a_1c_1}B^2_{a_2c_2}h^1_{a_3c_3}h^2_{a_4c_4}h^3_{a_5c_5}\int \frac{d^2\tau}{(\mathrm{Im}\,\tau)^5}\int\prod_{r=1}^{4} d^2v_r \prod_{r<s}(\chi_{rs})^{\frac{1}{2}k_r\cdot k_s} \quad (3.6)$$
$$\times\left(\sum_{r<s}\eta(v_{rs},\tau)A_{rs}\sum_{r<s}\bar{\eta}(v_{rs},\tau)\bar{A}_{rs} + \sum_{r<s}\hat{\Omega}(v_{rs},\tau)C_{rs}\right),$$

where the indices on A_{rs}, $\bar{A}_{rs}$ and C_{rs} have been suppressed for brevity,

$$\begin{aligned} A_{12} = {} & k_1^{a_2}(k_1+k_2)^b k_3^{b_3}k_4^{b_4}k_5^{b_5} t_8^{a_1ba_3b_3a_4b_4a_5b_5} \\ & - k_2^{a_1}(k_1+k_2)^b k_3^{b_3}k_4^{b_4}k_5^{b_5} t_8^{a_2ba_3b_3a_4b_4a_5b_5} \\ & - \delta^{a_1a_2}k_1^{b_1}k_2^{b_2}k_3^{b_3}k_4^{b_4}k_5^{b_5} t_8^{b_1b_2a_3b_3a_4b_4a_5b_5} \\ & - k_1\cdot k_2\, k_3^{b_3}k_4^{b_4}k_5^{b_5} t_8^{a_1a_2a_3b_3a_4b_4a_5b_5} \end{aligned} \quad (3.7)$$

and

$$C_{12} = -4\,\delta^{a_1c_2}k_2^{b_2}k_3^{b_3}k_4^{b_4}k_5^{b_5}k_1^{d_1}k_3^{d_3}k_4^{d_4}k_5^{d_5} t_8^{a_2b_2a_3b_3a_4b_4a_5b_5} t_8^{c_1d_1c_3d_3c_4d_4c_5d_5}. \quad (3.8)$$

$\bar{A}_{12}$ is the same as A_{12} but with c_r replacing a_r. The other A_{rs} and C_{rs} are similar but with the relevant permutations of the momenta and polarisation indices. The amplitude contains massless poles in $k_r\cdot k_s$ which originate from the v_{rs} integral over $|\eta(v_{rs},\tau)|^2$.

JHEP10(2008)043

There is no known way of explicitly evaluating the integrals over v_r and τ. However, it is still possible to expand them for small values of α'. Since we are only interested in the expansion to lowest order, there are only two types of integral that must be considered,[2]

$$K = \int \frac{d^2\tau}{(\mathrm{Im}\,\tau)^5} \int \prod_{r'=1}^{4} d^2 v_{r'} \prod_{r'<s'} (\chi_{r's'})^{\frac{1}{2}k_{r'}\cdot k_{s'}} \hat{\Omega}_{rs},$$
$$I_{rs} = \int \frac{d^2\tau}{(\mathrm{Im}\,\tau)^5} \int \prod_{r'=1}^{4} d^2 v_{r'} \prod_{r'<s'} (\chi_{r's'})^{\frac{1}{2}k_{r'}\cdot k_{s'}} |\eta_{rs}|^2, \tag{3.9}$$

where $\hat{\Omega}_{rs} \equiv \hat{\Omega}(v_{rs}, \tau)$ and similarly for η_{rs}. Since $\hat{\Omega}_{rs}$ is independent of v_{rs}, we write K without any subscripts. Using the definition of $\hat{\Omega}_{rs}$, the lowest-order expansion of K is easily found by setting $k_{r'} \cdot k_{s'}$ to zero for all r', s',

$$K|_{\alpha'^0} = -\frac{1}{2\pi} \int \frac{d^2\tau}{(\mathrm{Im}\,\tau)^2} = -\frac{1}{6}. \tag{3.10}$$

At lowest order the expansion of I_{rs} is a pole term and so it cannot be studied simply by setting $k_{r'} \cdot k_{s'} = 0$. However, it is clear that the pole originates from the corner of the integration region where $v_r \to v_s$. Then the pole can be extracted by writing $v_{rs} = |v|e^{i\theta}$ and integrating over a small circle around $v_{rs} = 0$. From the small v behaviour of χ_{rs} and η_{rs},

$$\chi(v,\tau) \sim 2\pi|v|, \qquad \eta(v,\tau) \sim -\frac{i}{2\pi v}, \tag{3.11}$$

it can easily be shown that in the small $k_r \cdot k_s$ limit,

$$I_{rs} \sim \frac{1}{k_r \cdot k_s} \cdot \frac{1}{\pi} \int \frac{d^2\tau}{(\mathrm{Im}\,\tau)^5} \int \prod_{r'=2}^{4} d^2 v_{r'} \prod_{\substack{r'<s' \\ 1\to 2}}{}' (\chi_{r's'})^{\frac{1}{2}k'_r\cdot k'_s}, \tag{3.12}$$

where the prime on the product indicates that $(r', s') = (1, 2)$ is not to be included, and $1 \to 2$ means that $v_{1'}$ is to be replaced by $v_{2'}$ everywhere within the product. The τ and $v_{r'}$ integrals are exactly those that appear in the four-graviton amplitude (2.1), which of course must be the case from unitarity, and so their low-energy expansion begins with $\pi/3$ as in (2.3), giving

$$I_{rs}|_{\alpha'^{-1}} = \frac{1}{6\alpha' k_r \cdot k_s}, \tag{3.13}$$

where α' has been reinstated using $2\alpha' = 1$. So the lowest-order expansion of the amplitude (3.6) is given by

$$A_{B^2h^3,t_8t_8}|_{\alpha'^4} = \frac{2^4\alpha'^4}{6} B^1_{a_1c_1} B^2_{a_2c_2} h^1_{a_3c_3} h^2_{a_4c_4} h^3_{a_5c_5} \sum_{r<s} \left(\frac{2}{k_r \cdot k_s} |A_{rs}|^2 - C_{rs} \right), \tag{3.14}$$

where again we have reinstated α'.

[2]The other possible integrals all vanish at lowest order in α'.

Unlike the five-graviton version in [1], various terms within the $|A_{12}|^2$, $|A_{1n}|^2$ and $|A_{2n}|^2$ parts, with $n = 3, 4, 5$, vanish due to identities such as $k_1^a k_1^b B_{2ab} = 0$ and $B_1^{ab} h_{1ab} = 0$. For example, of the twenty-one terms in $|A_{rs}|^2$, only fifteen survive in the $|A_{12}|^2$ part,

$$
\begin{aligned}
\frac{1}{k_1 \cdot k_2}\Big(& - 2k_{1e} k_1^{b_1} k_1^{d_1} B_1^{a_1}{}_f k_2^f B_2^{ec_1} - 2k_{1e} k_1^{b_1} B_1^{a_1}{}_f k_2^f k_2^{d_1} B_2^{ec_1} - 2k_{1e} k_1^{b_1} k_1^{c_1} B_1^{a_1}{}_f k_2^{d_1} B_2^{ef} \\
& - 2k_{1e} k_1^{d_1} B_1^{a_1}{}_f k_2^f k_2^{b_1} B_2^{ec_1} - 2k_{1e} B_1^{a_1}{}_f k_2^f k_2^{b_1} k_2^{d_1} B_2^{ec_1} - 2k_{1e} k_1^{c_1} B_1^{a_1}{}_f k_2^{b_1} k_2^{d_1} B_2^{ef} \\
& \quad + 2k_1^{b_1} k_1^{c_1} B_{1ef} k_2^{d_1} k_2^e B_2^{a_1 f} + 2k_1^{c_1} B_{1ef} k_2^{b_1} k_2^{d_1} k_2^e B_2^{a_1 f} + k_1^{a_1} k_1^{c_1} B_{1ef} k_2^{b_1} k_2^{d_1} B_2^{ef} \Big) \\
+ \Big(& - 2k_{1e} k_1^{b_1} B_1^{a_1 c_1} B_2^{ed_1} - 2k_{1e} B_1^{a_1 c_1} k_2^{b_1} B_2^{ed_1} + 2k_1^{b_1} B_{1e}{}^{c_1} k_2^e B_2^{a_1 d_1} \\
& \quad + 2B_{1e}{}^{c_1} k_2^e k_2^{b_1} B_2^{a_1 d_1} + 2k_1^{a_1} B_{1e}{}^{c_1} k_2^{b_1} B_2^{ed_1} + k_{1e} B_1^{a_1 c_1} k_2^e B_2^{b_1 d_1} \Big),
\end{aligned} \tag{3.15}
$$

which is all multiplied by $\frac{2^4 \alpha'^4}{3} t_{8\, a_1 b_1 \ldots} t_{8\, c_1 d_1 \ldots} k_3^{a_2} k_3^{c_2} h_1^{b_2 d_2} k_4^{a_3} k_4^{c_3} h_2^{b_3 d_3} k_5^{a_4} k_5^{c_4} h_3^{b_4 d_4}$. Similarly, for the $|A_{13}|^2$ part only seventeen terms remain,

$$
\begin{aligned}
\frac{1}{k_1 \cdot k_3}\Big(& + k_{1e} k_{1f} k_1^{b_1} k_1^{d_1} B_1^{a_1 c_1} h_1^{ef} + 2k_{1e} k_{1f} k_1^{b_1} B_1^{a_1 c_1} k_3^{d_1} h_1^{ef} - 2k_{1e} k_1^{b_1} k_1^{d_1} B_1^{a_1}{}_f k_3^f h_1^{ec_1} \\
& - 2k_{1e} k_1^{b_1} B_1^{a_1}{}_f k_3^f k_3^{d_1} h_1^{ec_1} - 2k_{1e} k_1^{b_1} k_1^{c_1} B_1^{a_1}{}_f k_3^{d_1} h_1^{ef} + k_{1e} k_{1f} B_1^{a_1 c_1} k_3^{b_1} k_3^{d_1} h_1^{ef} \\
& - 2k_{1e} k_1^{d_1} B_1^{a_1}{}_f k_3^f k_3^{b_1} h_1^{ec_1} - 2k_{1e} B_1^{a_1}{}_f k_3^f k_3^{b_1} k_3^{d_1} h_1^{ec_1} - 2k_{1e} k_1^{c_1} B_1^{a_1}{}_f k_3^{b_1} k_3^{d_1} h_1^{ef} \\
& \qquad + 2k_1^{b_1} k_1^{c_1} B_{1ef} k_3^{d_1} k_3^e h_1^{a_1 f} + 2k_1^{c_1} B_{1ef} k_3^{b_1} k_3^{d_1} k_3^e h_1^{a_1 f} \Big) \\
+ \Big(& - 2k_{1e} k_1^{b_1} B_1^{a_1 c_1} h_1^{ed_1} - 2k_{1e} B_1^{a_1 c_1} k_3^{b_1} h_1^{ed_1} + 2k_1^{b_1} B_{1e}{}^{c_1} k_3^e h_1^{a_1 d_1} \\
& \quad + 2B_{1e}{}^{c_1} k_3^e k_3^{b_1} h_1^{a_1 d_1} + 2k_1^{a_1} B_{1e}{}^{c_1} k_3^{b_1} h_1^{ed_1} + k_{1e} B_1^{a_1 c_1} k_3^e h_1^{b_1 d_1} \Big),
\end{aligned} \tag{3.16}
$$

which is all multiplied by $\frac{2^4 \alpha'^4}{3} t_{8\, a_1 b_1 \ldots} t_{8\, c_1 d_1 \ldots} k_2^{a_2} k_2^{c_2} B_2^{b_2 d_2} k_4^{a_3} k_4^{c_3} h_2^{b_3 d_3} k_5^{a_4} k_5^{c_4} h_3^{b_4 d_4}$. These expressions will be important for matching with the field theory diagrams in the next section.

4. Consequences for the effective action

The expanded amplitude in (3.14) can now be compared with the same amplitude calculated from known quartic terms in the effective action, such as R^4 and $R^2(\nabla H)^2$. These terms, however, will not account for the full amplitude, and the remainder will require new $H^2 R^3$ terms.

At lowest order the effective action consists of the usual supergravity terms, which in Einstein frame are given by

$$
S_{\alpha'^0} = \int d^{10}x \sqrt{-g} \left(R - \frac{1}{12} e^{-\phi} H^2 - \frac{1}{2} (\partial \phi)^2 \right). \tag{4.1}
$$

Einstein frame is used to avoid mixing between the dilaton and graviton propagators. It is important to include the dilaton since there are diagrams where a dilaton propagates as an intermediate particle. As mentioned in section 2, the first correction to $S_{\alpha'^0}$ occurs at order α'^3. For our purposes, only the one-loop correction is relevant,

$$S_{\alpha'^3,1-\text{loop}} = \frac{2\pi^2}{3}\alpha'^3 \int d^{10}x\sqrt{-g}\, e^{\phi/2}\, t_8^{a_1b_1\cdots} t_8^{c_1d_1\cdots} \bar{R}_{a_1b_1c_1d_1}\cdots \bar{R}_{a_4b_4c_4d_4}, \tag{4.2}$$

where $\bar{R}_{abcd}$ is given in (2.7). In particular, for matching with the B^2h^3 amplitude, only the $t_8t_8R^4$, $t_8t_8R^2(\nabla H)^2$ and $t_8t_8R^3(\nabla\nabla\phi)$ terms are required.

4.1 Expansion of various tensors

Before we can expand the terms in $S_{\alpha'^0}$ and $S_{\alpha'^3}$, we need the expansions of the various fields and tensors involved. Consider a small fluctuation of the metric about the Minkowski metric, $g_{ab} = \eta_{ab} + \kappa h_{ab}$, where κ is presumed small. In subsequent expressions we will drop factors of κ since they can easily be reinstated. The expansions of the Riemann tensor, the Ricci scalar and the t_8 tensor were given in [1] and we refer the reader there for details.

At most we require the expansion of the Riemann tensor, R_{abcd}, to second order in h. As explained in [1], since we are only concerned with five-point amplitudes, a Riemann tensor expanded to second order is guaranteed to be multiplied by a tensor which is antisymmetric in $a \leftrightarrow b$ and $c \leftrightarrow d$, and symmetric in $(a, b) \leftrightarrow (c, d)$. With this understanding, the expansion to second order simplifies to

$$R_{abcd} = 2\partial_a\partial_c h_{bd} + \partial_a h_c{}^e \partial_d h_{be} + \partial_a h_c{}^e \partial_b h_{de} - 2\partial^e h_{ac}\partial_b h_{de} + \frac{1}{2}\partial^e h_{ac}\partial_e h_{bd}. \tag{4.3}$$

The expansion of the Ricci scalar begins

$$\begin{aligned} R = {} & \Box h - \partial_a\partial_b h^{ab} \\ & - h^{ab}(\Box h_{ab} + \partial_a\partial_b h - 2\partial_a\partial^c h_{bc}) \\ & - \frac{3}{4}\partial_a h_{bc}\partial^a h^{bc} + \frac{1}{2}\partial_a h_{bc}\partial^b h^{ac} + \partial^a h_{ab}\partial_c h^{bc} - \partial^a h_{ab}\partial^b h + \frac{1}{4}\partial^a h\partial_a h, \end{aligned} \tag{4.4}$$

where $h = h^a{}_a$. Although we actually need the expansion to third order in h, we do not need the explicit expression and so there is no need to write it here.

Since the t_8 tensor is formed from products of the metric, it is important to also consider its expansion. As for the Riemann tensor, since whenever t_8 is expanded to first order it is always multiplied by tensors which are symmetric under, for example, $(a, b) \leftrightarrow (c, d)$, the expansion reduces to

$$t_8^{abcdefgh} = \underline{t}_8^{abcdefgh} - 2(h_i{}^a \underline{t}_8^{ibcdefgh} + h_i{}^b \underline{t}_8^{aicdefgh}), \tag{4.5}$$

where t_8 is formed out of products of the curved metric, g, and $\underline{t}_8$ is the equivalent expression formed out of the Minkowski metric, η.

The expansion of H^2 is achieved using

$$H^2 \equiv g^{ad}g^{be}g^{cf}H_{abc}H_{def} \equiv 3^2 g^{ad}g^{be}g^{cf}\nabla_{[a}B_{bc]}\nabla_{[d}B_{ef]} \tag{4.6}$$

and remembering that it is also necessary to expand the covariant derivatives. It is readily found that, up to first order in h,

$$2H^2 = \partial_a B_{bc}(\partial^a B^{bc} + 2\partial^b B^{ca}) - h^{ad}(\partial_a B_{bc}\partial_d B^{bc} - 4\partial_a B_{bc}\partial^b B_d{}^c + 2\partial_b B_{ac}\partial^b B_d{}^c - 2\partial_b B_{ac}\partial^c B_d{}^b). \quad (4.7)$$

Since we need the expansion of $S_{\alpha'^3}$ up to terms involving five fields, we require the expansion of $\nabla_{[a}H_{b]cd}$ up to first order in h. The zeroth order contribution is trivial and the first order terms originate from expanding the Christoffel symbols within the derivative. When $\nabla_{[a}H_{b]cd}$ is expanded to first order it is assured of being multiplied by an expression which is manifestly antisymmetric in $a \leftrightarrow b$, antisymmetric in $c \leftrightarrow d$, and *antisymmetric* in $(a, b) \leftrightarrow (c, d)$.[3] With the understanding that $\nabla_{[a}H_{b]cd}$ is multiplied by a tensor with such symmetries, the expansion simplifies and, up to first order in h, can be written as

$$\frac{1}{2}\nabla_{[a}H_{b]cd} = \partial_a\partial_c B_{db} + \partial_a h_c{}^e\partial_d B_{be} + \partial_a h_c{}^e\partial_b B_{ed} - \partial^e h_{ac}\partial_b B_{ed} - \partial_a h_c{}^e\partial_e B_{bd} + \frac{1}{2}\partial^e h_{ac}\partial_e B_{bd}, \quad (4.8)$$

which, although similar to the expansion of the Riemann tensor, differs due to the different symmetries of h_{ab} and B_{ab}, and the missing factor of two in the zeroth order term.

4.2 Propagators

From (4.1) we can derive the propagators for the graviton, the $NS \otimes NS$ two-form and the dilaton. Since the dilaton is a scalar, its propagator is simply $D = 1/k^2$. For the graviton we consider the Einstein-Hilbert term which, after dropping total derivatives, is given up to second order by,

$$S_{\alpha^0,R} = \frac{1}{4}\int d^{10}x\,(\partial_a h_{bc}\partial^a h^{bc} - \partial_a h\partial^a h + 2\partial_a h\partial_b h^{ab} - 2\partial_a h_{bc}\partial^b h^{ac}), \quad (4.9)$$

which is invariant under the gauge transformation

$$h_{ab} \to h_{ab} + \partial_a\zeta_b + \partial_b\zeta_a, \quad (4.10)$$

where ζ_a is an arbitrary one-form field. After fixing the gauge invariance using the de Donder gauge, $\partial^a h_{ab} = \frac{1}{2}\partial_b h$, the graviton propagator can easily be shown to be

$$D_{ab,cd} = \frac{\eta_{ac}\eta_{bd} + \eta_{ad}\eta_{bc} - \frac{1}{4}\eta_{ab}\eta_{cd}}{k^2}. \quad (4.11)$$

To find the propagator for the two-form we first fix the gauge invariance, $B_{ab} \to B_{ab} + \nabla_{[a}\zeta_{b]}$, by adding the gauge fixing term $\lambda\partial_a B^{ac}\partial_b B^b{}_c$ to the action. Then, after removing a total derivative, the part of (4.1) quadratic in B_{ab} becomes

$$S_{\alpha'^0,H^2} = -\frac{1}{24}\int d^{10}x(\partial_a B_{bc}\partial^a B^{bc} - 2B_{bc}\partial_a\partial^b B^{ca} + 24\lambda B^{ac}\partial_a\partial_b B^b{}_c). \quad (4.12)$$

[3]This antisymmetry follows since, from the Bianchi identity, $\nabla_{[a}H_{b]cd} = -\nabla_{[c}H_{d]ab}$.

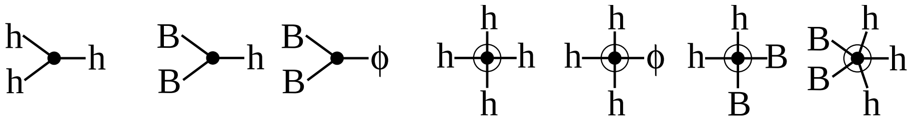


Figure 1: Field theory vertices relevant for the B^2h^3 amplitude. From left to right: a three-vertex from R, two three-vertices from $e^{-\phi}H^2$, and three four-vertices and a five-vertex from $\bar{R}^4$.

Now we write the integrand as $B_{ab}V^{abcd}B_{cd}$ and try to invert V^{abcd}, by which we mean solve $D'_{ab,cd}V^{cd}{}_{ef} = \eta_{[a|e|}\eta_{b]f}$. This can only be achieved for the particular choice $\lambda = -\frac{1}{12}$, after which we find

$$D'_{ab,cd} = \frac{\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc}}{k^2}. \tag{4.13}$$

4.3 Evaluation of diagrams

The expansions of $S_{\alpha'^0}$ and $S_{\alpha'^3}$ lead to several three-, four- and five-vertices which appear in B^2h^3 diagrams. Firstly, the Einstein-Hilbert term contains the usual three-graviton vertex. Further, the kinetic term for the $NS \otimes NS$ two-form, $e^{-\phi}H^2 = H^2 - \phi H^2 + \ldots$, gives a BBh three-vertex from the expansion of the first term, and a $BB\phi$ three-vertex from the second term. The quartic one-loop term, $S_{\alpha'^3}$, leads to three relevant four-vertices: a four-graviton vertex, a three-graviton and one-dilaton vertex, and a two-graviton and two-B-field vertex. Finally, the $R^2(\nabla H)^2$ term in $S_{\alpha'^3}$ generates a $BBhhh$ five-vertex. These vertices are shown in figure 1 where a vertex surrounded by a circle originates from the one-loop $S_{\alpha'^3}$ term, whereas a vertex without a circle originates from the tree-level $S_{\alpha'^0}$ term.

Unlike the five-graviton amplitude in [1] where there were only two field theory diagrams to calculate, there are now five separate diagrams to evaluate (figure 2). Diagrams (a) and (b) both contain internal states carrying momenta $\sqrt{-2k_1 \cdot k_2}$, the first with an intermediate graviton and the second with an intermediate dilaton. Diagrams (c) and (d) are similar: the first contains an intermediate B-field carrying momenta $\sqrt{-2k_r \cdot k_s}$ and the second contains an intermediate graviton carrying momenta $\sqrt{-2k_s \cdot k_{s'}}$, where $r \in \{1, 2\}$ and $s, s' \in \{3, 4, 5\}$. Diagram (e) is a contact diagram formed from a single five-vertex. All these diagrams must be evaluated and subtracted from the B^2h^3 amplitude, before the remaining terms can be covariantised.

Diagram (a) consists of a BBh three-vertex connected via a graviton propagator to a four-graviton one-loop vertex and can be evaluated as follows. The three-vertex is found from (4.7) and first must be contracted into the two on-shell B-fields, B_1 and B_2. The two free indices are then contracted with the graviton propagator (4.11), before multiplying by the four-graviton vertex derived from $t_8t_8R^4$. Finally, the remaining external legs contract

JHEP10(2008)043

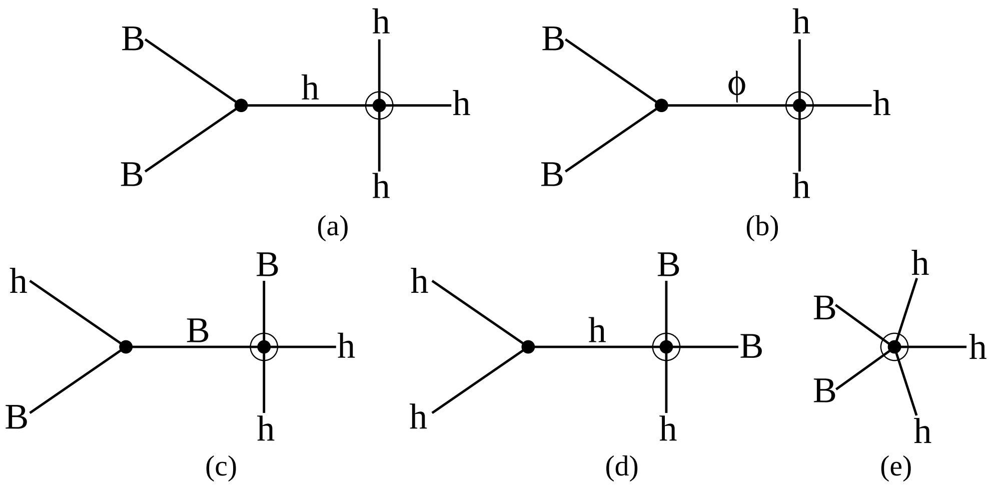
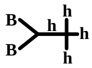


Figure 2: Field theory diagrams contributing to the B^2h^3 amplitude.

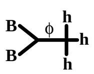
into the three gravitons, h_3, h_4 and h_5. After some work, the result simplifies to



$$
\begin{aligned}
&= t_{8\,a_1b_1\ldots}t_{8\,c_1d_1\ldots}k_3^{a_2}k_3^{c_2}h_1^{b_2d_2}\,k_4^{a_3}k_4^{c_3}h_2^{b_3d_3}\,k_5^{a_4}k_5^{c_4}h_3^{b_4d_4}\\
&\times\Bigg(\frac{1}{k_1\cdot k_2}\Bigg(-2k_{1e}k_1^{b_1}k_1^{d_1}B_1^{a_1}{}_f k_2^f B_2^{ec_1}-2k_{1e}k_1^{b_1}B_1^{a_1}{}_f k_2^f k_2^{d_1}B_2^{ec_1}\\
&\qquad -2k_{1e}k_1^{b_1}k_1^{c_1}B_1^{a_1}{}_f k_2^{d_1}B_2^{ef}-2k_{1e}k_1^{d_1}B_1^{a_1}{}_f k_2^f k_2^{b_1}B_2^{ec_1}\\
&\qquad -2k_{1e}B_1^{a_1}{}_f k_2^f k_2^{b_1}k_2^{d_1}B_2^{ec_1}-2k_{1e}k_1^{c_1}B_1^{a_1}{}_f k_2^{b_1}k_2^{d_1}B_2^{ef}\\
&\qquad +2k_1^{b_1}k_1^{c_1}B_{1ef}k_2^{d_1}k_2^e B_2^{a_1f}+2k_1^{c_1}B_{1ef}k_2^{b_1}k_2^{d_1}k_2^e B_2^{a_1f}\\
&\qquad +k_1^{a_1}k_1^{c_1}B_{1ef}k_2^{b_1}k_2^{d_1}B_2^{ef}\\
&\qquad -\frac{3}{4}(k_1+k_2)^{a_1}(k_1+k_2)^{c_1}k_{1e}B_{1fg}k_2^g B_2^{fe}\eta^{b_1d_1}\Bigg)\\
&\quad +(k_1+k_2)^{a_1}(k_1+k_2)^{c_1}\left(2B_{1e}{}^{(d_1}B_2^{b_1)e}+\frac{3}{8}B_{1ef}B_2^{ef}\eta^{b_1d_1}\right)\Bigg),
\end{aligned}
\tag{4.14}
$$

where all terms apart from those in the last line are poles. The penultimate line is a pole containing an η^{ab} factor. Since there are no such terms in the B^2h^3 amplitude, this must cancel with an equivalent term from another diagram.

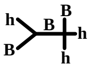
Diagram (b) is similar in spirit to diagram (a), but with a dilaton as the intermediate particle. As such, the simpler dilaton propagator is used to contract the two vertices. The three-vertex now originates from the ϕH^2 term in $S_{\alpha'^0}$. The diagram is easily evaluated as



$$
\begin{aligned}
&= t_{8\,a_1b_1\ldots}t_{8\,c_1d_1\ldots}k_3^{a_2}k_3^{c_2}h_1^{b_2d_2}\,k_4^{a_3}k_4^{c_3}h_2^{b_3d_3}\,k_5^{a_4}k_5^{c_4}h_3^{b_4d_4}\\
&\quad\times(k_1+k_2)^{a_1}(k_1+k_2)^{c_1}\eta^{b_1d_1}\left(\frac{3}{4k_1\cdot k_2}k_{1e}B_{1fg}k_2^g B_2^{fe}-\frac{3}{8}B_{1ef}B_2^{ef}\right).
\end{aligned}
\tag{4.15}
$$

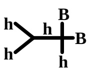
The pole term has exactly the correct form to cancel the pole containing η^{ab} in the previous diagram, and the second term exactly cancels the final term in (4.14). So the sum of diagrams (a) and (b) is given by (4.14), but without the two terms containing η^{ab} factors.

Diagram (c) accounts for the poles where a B-field and a graviton are separated from the other particles. Consider the particular case where the separated B-field is B_1 with momentum k_1 and the separated graviton is h_1 with momentum k_3. We start from the BBh three-vertex given by (4.7) and contract B_1 and h_1 into two of the legs. The remaining leg is now connected by the two-form propagator (4.13) to a B^2h^2 four-vertex from the expansion of $t_8t_8\bar{R}^4$. Finally, the remaining three legs are contracted into B_2, h_2 and h_3. After simplifying the result we obtain



$$
\begin{aligned}
&= t_{8\,a_1b_1\ldots}t_{8\,c_1d_1\ldots}k_2^{a_2}k_2^{c_2}B_2^{b_2d_2}\,k_4^{a_3}k_4^{c_3}h_2^{b_3d_3}\,k_5^{a_4}k_5^{c_4}h_3^{b_4d_4}\\
&\quad\times\Bigg(\frac{1}{k_1\cdot k_3}\Big(+k_{1e}k_{1f}k_1^{b_1}k_1^{d_1}B_1^{a_1c_1}h_1^{ef}+2k_{1e}k_{1f}k_1^{b_1}B_1^{a_1c_1}k_3^{d_1}h_1^{ef}\\
&\qquad-2k_{1e}k_1^{b_1}k_1^{d_1}B_1^{a_1}{}_fk_3^fh_1^{ec_1}-2k_{1e}k_1^{b_1}B_1^{a_1}{}_fk_3^fk_3^{d_1}h_1^{ec_1}\\
&\qquad-2k_{1e}k_1^{b_1}k_1^{c_1}B_1^{a_1}{}_fk_3^{d_1}h_1^{ef}+k_{1e}k_{1f}B_1^{a_1c_1}k_3^{b_1}k_3^{d_1}h_1^{ef}\\
&\qquad-2k_{1e}k_1^{d_1}B_1^{a_1}{}_fk_3^fk_3^{b_1}h_1^{ec_1}-2k_{1e}B_1^{a_1}{}_fk_3^fk_3^{b_1}k_3^{d_1}h_1^{ec_1}\\
&\qquad-2k_{1e}k_1^{c_1}B_1^{a_1}{}_fk_3^{b_1}k_3^{d_1}h_1^{ef}+2k_1^{b_1}k_1^{c_1}B_{1ef}k_3^{d_1}k_3^eh_1^{a_1f}\\
&\qquad+2k_1^{c_1}B_{1ef}k_3^{b_1}k_3^{d_1}k_3^eh_1^{a_1f}\Big)\\
&\quad+2(k_1+k_3)^{a_1}(k_1+k_3)^{c_1}B_{1e}{}^{[d_1}h_1^{b_1]e}\Bigg).
\end{aligned}
\tag{4.16}
$$

Diagram (d) is almost identical to the pole diagram calculated in [1], which itself was derived from a similar amplitude in [17], the only difference being the four-vertex. This difference, however, is minimal since R_{abcd} and $\nabla_{[a}H_{b]cd}$ expanded to lowest order both have the form $2\partial_a\partial_cX_{db}$, where X is h and B respectively. As such, we can simply take the result in [1] and replace the relevant gravitons by B-fields. Let the two gravitons which are to the left of the propagator be h_1 and h_2 with momenta k_3 and k_4 respectively. Then

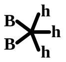


$$
\begin{aligned}
&= t_{8\,a_1b_1\ldots}t_{8\,c_1d_1\ldots}k_1^{a_2}k_1^{c_2}B_1^{b_2d_2}\,k_2^{a_3}k_2^{c_3}B_2^{b_3d_3}\,k_5^{a_4}k_5^{c_4}h_3^{b_4d_4}\\
&\quad\times\Bigg(\frac{1}{k_3\cdot k_4}\Big(+k_{3e}k_{3f}k_3^{b_1}k_3^{d_1}h_1^{a_1c_1}h_2^{ef}+2k_{3e}k_{3f}k_3^{b_1}h_1^{a_1c_1}k_4^{d_1}h_2^{ef}\\
&\qquad-2k_{3e}k_3^{b_1}k_3^{d_1}h_1^{a_1}{}_fk_4^fh_2^{ec_1}-2k_{3e}k_3^{b_1}h_1^{a_1}{}_fk_4^fk_4^{d_1}h_2^{ec_1}\\
&\qquad-2k_{3e}k_3^{b_1}k_3^{c_1}h_1^{a_1}{}_fk_4^{d_1}h_2^{ef}+k_{3e}k_{3f}h_1^{a_1c_1}k_4^{b_1}k_4^{d_1}h_2^{ef}\\
&\qquad-2k_{3e}k_3^{d_1}h_1^{a_1}{}_fk_4^fk_4^{b_1}h_2^{ec_1}-2k_{3e}h_1^{a_1}{}_fk_4^fk_4^{b_1}k_4^{d_1}h_2^{ec_1}\\
&\qquad-2k_{3e}k_3^{c_1}h_1^{a_1}{}_fk_4^{b_1}k_4^{d_1}h_2^{ef}+k_3^{b_1}k_3^{d_1}h_{1ef}k_4^ek_4^fh_2^{a_1c_1}\\
&\qquad+2k_3^{b_1}h_{1ef}k_4^ek_4^fk_4^{d_1}h_2^{a_1c_1}+2k_3^{b_1}k_3^{c_1}h_{1ef}k_4^{d_1}k_4^eh_2^{a_1f}\\
&\qquad+h_{1ef}k_4^{b_1}k_4^{d_1}k_4^ek_4^fh_2^{a_1c_1}+2k_3^{c_1}h_{1ef}k_4^{b_1}k_4^{d_1}k_4^eh_2^{a_1f}\\
&\qquad+k_3^{a_1}k_3^{c_1}h_{1ef}k_4^{b_1}k_4^{d_1}h_2^{ef}\Big)\\
&\quad+2(k_3+k_4)^{a_1}(k_3+k_4)^{c_1}h_{1e}{}^{(d_1}h_2^{b_1)e}\Bigg).
\end{aligned}
\tag{4.17}
$$

Finally we calculate diagram (e), which involves expanding the $R^2(\nabla H)^2$ term from the one-loop $S_{\alpha'^3}$ action,

$$\int d^{10}x\sqrt{-g}\, t_8^{a_1b_1\cdots} t_8^{c_1d_1\cdots} \nabla_{[a_1}H_{b_1]c_1d_1} \nabla_{[a_2}H_{b_2]c_2d_2} R_{a_3b_3c_3d_3} R_{a_4b_4c_4d_4}, \tag{4.18}$$

to third order in h and contracting all legs into the on-shell external states. Since at lowest order $S_{\alpha'^3}$ contains two gravitons, it is necessary to expand to one order higher than leading-order. The fifth graviton cannot originate from $\sqrt{-g} \approx 1 + \frac{1}{2}h^a{}_a$ since the external gravitons are traceless. However, it can originate either from a t_8 tensor, from a Riemann tensor or from a $\nabla_{[a}H_{b]cd}$ factor. After a slightly involved calculation which uses (4.5), (4.3) and (4.8), we find



$$\begin{aligned} = {} & 2^5 t_{8\,a_1b_1\ldots} t_{8\,c_1d_1\ldots} k_2^{a_2} k_2^{c_2} B_2^{b_2d_2}\, k_4^{a_3} k_4^{c_3} h_2^{b_3d_3}\, k_5^{a_4} k_5^{c_4} h_3^{b_4d_4} \\ & \times \Big(+ k_1^{d_1} B_1^{b_1}{}_e k_3^{a_1} h_1^{c_1e} + k_1^{b_1} B_{1e}{}^{d_1} k_3^{a_1} h_1^{c_1e} - k_1^{b_1} B_{1e}{}^{d_1} k_3^e h_1^{a_1c_1} \\ & \quad - k_{1e} B_1^{b_1d_1} k_3^{a_1} h_1^{c_1e} + \frac{1}{2} k_{1e} B_1^{b_1d_1} k_3^e h_1^{a_1c_1} - k_{1e} k_1^{c_1} B_1^{b_1d_1} h_1^{a_1e} \\ & \quad - k_1^{a_1} k_1^{c_1} B_{1e}{}^{d_1} h_1^{b_1e} \Big) \\ & + 2^4 t_{8\,a_1b_1\ldots} t_{8\,c_1d_1\ldots} k_1^{a_2} k_1^{c_2} B_1^{b_2d_2}\, k_2^{a_3} k_2^{c_3} B_2^{b_3d_3}\, k_5^{a_4} k_5^{c_4} h_3^{b_4d_4} \\ & \quad \times \Big(+ k_3^{d_1} h_1^{b_1}{}_e k_4^{a_1} h_2^{c_1e} + k_3^{b_1} h_{1e}{}^{d_1} k_4^{a_1} h_2^{c_1e} - k_3^{b_1} h_{1e}{}^{d_1} k_4^e h_2^{a_1c_1} \\ & \quad\quad - k_{3e} h_1^{b_1d_1} k_4^{a_1} h_2^{c_1e} + \frac{1}{2} k_{3e} h_1^{b_1d_1} k_4^e h_2^{a_1c_1} - k_{3e} k_3^{c_1} h_1^{b_1d_1} h_2^{a_1e} \\ & \quad\quad - k_3^{a_1} k_3^{c_1} h_{1e}{}^{d_1} h_2^{b_1e} - h_1^{b_1}{}_e k_4^{a_1} k_4^{c_1} h_2^{d_1e} + h_1^{d_1}{}_e k_4^e k_4^{a_1} h_2^{b_1c_1} \Big) \\ & + \text{all permutations of } (B_1,\, B_2) \text{ and of } (h_1,\, h_2,\, h_3). \end{aligned} \tag{4.19}$$

4.4 New H^2R^3 terms

All the above diagrams need to be subtracted from (3.14), before the remainder can be covariantised to discover new H^2R^3 terms. The matching of the pole terms is guaranteed from unitarity and indeed it is readily seen that the poles in the sum of (4.14) and (4.15) match with the poles in (3.15), that the poles in (4.16) match with the poles in (3.16), and that the poles in (4.17) match with the poles in the $|A_{rs}|^2$ part of (3.14) for $r, s = 3, 4, 5$.

This leaves the non-poles. In the amplitude these arise from both the $|A_{rs}|^2$ and C_{rs} terms. In the field theory they originate from both the non-pole terms in diagrams (a) to (d) and from the entirety of the contact diagram. Consider first terms where two gravitons are singled out. As explained in [1], it is always possible to single out such terms even for the non-poles. Then the non-poles in (3.14) can easily be seen to match the non-poles from (4.17) and the second half of (4.19). The matching is practically identical to that for the lowest-order expansion of the five-graviton amplitude in [1].

Next consider terms where a B-field and a graviton are singled out. For concreteness assume B_1 and h_1 to be the separated fields. Then we need to compare the final two lines in (3.16) and the C_{13} term with the final line in (4.16) and the first half of (4.19). After doing so, not all terms cancel; the remainder are given by

$$2t_{8\,a_1b_1\ldots}t_{8\,c_1d_1\ldots}k_2^{a_2}k_2^{c_2}B_2^{b_2d_2}k_4^{a_3}k_4^{c_3}h_2^{b_3d_3}k_5^{a_4}k_5^{c_4}h_3^{b_4d_4}\big(B_{1e}{}^{c_1}k_3^{e}k_3^{b_1}h_1^{a_1d_1}-B_{1e}{}^{a_1}k_3^{b_1}k_3^{d_1}h_1^{c_1e}\big). \tag{4.20}$$

Such terms are potentially problematic since they cannot be covariantised into new effective action terms. An attempt to do so would generate terms of the form $t_8t_8B(\nabla H)R^3$, but such terms cannot appear since they are not gauge-invariant. So there must be a mechanism whereby these extra terms cancel against other terms.

Finally consider terms where the two B-fields are singled out. This involves comparing the final two lines in (3.15) and the C_{12} term with the penultimate term in (4.14); the contact diagram (4.19) now makes no contribution. After cancelling the field theory diagrams, the remaining terms are given by

$$\begin{aligned}&2t_{8\,a_1b_1\ldots}t_{8\,c_1d_1\ldots}k_3^{a_2}k_3^{c_2}h_1^{b_2d_2}k_4^{a_3}k_4^{c_3}h_2^{b_3d_3}k_5^{a_4}k_5^{c_4}h_3^{b_4d_4}\\ &\quad\times\Big(+B_{1e}{}^{c_1}k_2^{e}k_2^{b_1}B_2^{a_1d_1}-B_{1e}{}^{a_1}k_2^{b_1}k_2^{d_1}B_2^{c_1e}-k_{1e}k_1^{b_1}B_1^{a_1c_1}B_2^{ed_1}\\ &\qquad-k_1^{b_1}k_1^{d_1}B_{1e}{}^{a_1}B_2^{c_1e}-k_{1e}B_1^{a_1c_1}k_2^{b_1}B_2^{ed_1}+k_1^{b_1}B_{1e}{}^{c_1}k_2^{e}B_2^{a_1d_1}\\ &\qquad+k_1^{a_1}B_{1e}{}^{c_1}k_2^{b_1}B_2^{ed_1}-k_1^{d_1}B_{1e}{}^{a_1}k_2^{b_1}B_2^{c_1e}+\frac{1}{2}k_{1e}B_1^{a_1c_1}k_2^{e}B_2^{b_1d_1}\Big).\end{aligned} \tag{4.21}$$

Not all of these terms can be generated from new terms in the effective action: those terms with 'naked' B-fields, i.e. those without any momenta multiplying them such as the B_1 in the first term, cannot be covariantised in a gauge-invariant manner. The resolution is that the first four terms in (4.21) and the two terms in (4.20) actually cancel when summed over all permutations of the external states. The proof of this uses an identity between four $\bar{t}_{10}$ tensors which is shown in the appendix of [1]. Consider

$$\begin{aligned}&(\bar{t}_{10}^{ABa_1b_1a_2b_2a_3b_3a_4b_4}+\bar{t}_{10}^{ABa_2b_2a_1b_1a_3b_3a_4b_4}+\bar{t}_{10}^{ABa_3b_3a_1b_1a_2b_2a_4b_4}+\bar{t}_{10}^{ABa_4b_4a_1b_1a_2b_2a_3b_3})\\ &\quad\times t_8^{c_1d_1c_2d_2c_3d_3c_4d_4}B_{1AB}k_{2a_1}k_{2c_1}B_{2b_1d_1}k_{3a_2}k_{3c_2}h_{1b_2d_2}k_{4a_3}k_{4c_3}h_{2b_3d_3}k_{5a_4}k_{5c_4}h_{3b_4d_4},\end{aligned} \tag{4.22}$$

which vanishes due to identity (A.4) in [1]. By expanding each of the $\bar{t}_{10}$ tensors as a sum of four t_8 tensors as in (A.2) in [1], it is easy to show that this gives the first two terms in (4.21) and the two terms in (4.20) summed over all permutations of the gravitons. The final two 'naked' B-terms in (4.21) cancel with the $B_1 \leftrightarrow B_2$ equivalent of (4.20), which is shown by considering the above identity but with B_1 and B_2 interchanged.

This leaves the final five terms in (4.21) and we can now ask what term or terms in the effective action might give rise to them. By considering

$$\alpha'^3\int d^{10}x\sqrt{-g}\,t_8^{a_1b_1a_2b_2a_3b_3a_4b_4}t_8^{c_1d_1c_2d_2c_3d_3c_4d_4}H_{a_1c_1e}H_{b_1d_1}{}^{e}R_{a_2b_2c_2d_2}R_{a_3b_3c_3d_3}R_{a_4b_4c_4d_4} \tag{4.23}$$

it is easy to see that, at lowest order in h, exactly these five extra terms are generated and this then is our required new term to account for the B^2h^3 one-loop amplitude.

This term agrees with the conjecture in [18] (see their (2.11)), but disagrees with the covariant RNS calculation in [3], where it is claimed that the t_8t_8 part of the H^2R^3

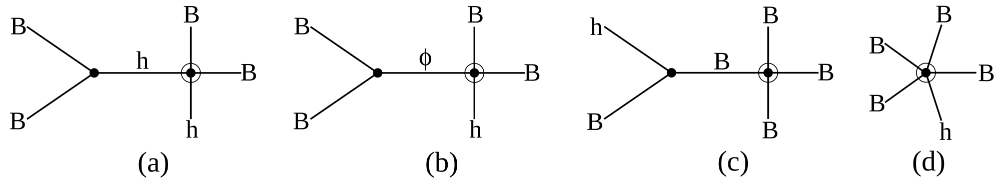


Figure 3: Field theory diagrams contributing to the B^4h amplitude: (a)-(c) pole diagrams, and (d) a contact diagram.

term is as above but with the H^2 part replaced by $H_{a_1b_1e}H_{c_1d_1}{}^e$. We believe the difference arises because [3], as indeed acknowledged there, makes no attempt to subtract field theory diagrams due to known quartic terms in the effective action. It is also worth noting that our result disagrees with the light-cone gauge GS calculation in [19].

5. The B^4h amplitude

It is relatively straightforward to extend the whole of the previous analysis to the B^4h amplitude. In calculating the B^2h^3 amplitude in section 3 (and in the five-graviton case in [1]) at most two vertex operators give ∂X^a and $\bar{\partial} X^a$ terms, with the remaining three operators giving $R^{ab}k^b\tilde{R}^{cd}k^d$ factors. These three factors simply lead to $k_bk_dX_{ac}$ factors in the amplitude, where X is either h or B, and so the difference between B^2h^3 and B^4h is minimal. Let the four $NS\otimes NS$ two-forms be B_1, B_2, B_3, B_4 with momenta k_1, k_2, k_3, k_4 respectively, and let the graviton be labelled h_1 and have momentum k_5. Then the B^4h amplitude is given by (3.6) but with $h^1h^2h^3$ replaced by $B^3B^4h^1$. Similarly, the lowest-order expansion is given by the same replacement in (3.14).

There is also little difference between the effective action diagrams for B^2h^3 and B^4h. This follows since, as explained in [1], all diagrams, even the contact diagram, contain exactly two particles which play a privileged rôle. The remainder merely act as spectators, appearing as R_{abcd} or $\nabla_{[a}H_{b]cd}$ factors expanded to leading order. So there is little point explicitly evaluating the new diagrams. However, for completeness, the relevant diagrams are shown in figure 3 where several new vertices are required: the four-vertex from the lowest-order expansion of the $(\nabla H)^4$ term in (4.2), the equivalent vertex from the $R\phi(\nabla H)^2$ term, and the five-vertex from the expansion of $(\nabla H)^4$ to first order in h.

The poles in all channels match in an identical manner to that for the B^2h^3 case and the 'naked' B-fields cancel using the same $\bar{t}_{10}$ identity. This leaves the same five $kBkB$ terms as for B^2h^3, but now with two of the $k_ak_ch_{bd}$ factors replaced by $k_ak_cB_{bd}$. So the new term required in the effective action is given by

$$\alpha'^3\int d^{10}x\sqrt{-g}\,t_8^{a_1b_1a_2b_2a_3b_3a_4b_4}t_8^{c_1d_1c_2d_2c_3d_3c_4d_4}$$
$$\times H_{a_1c_1e}H_{b_1d_1}{}^e\nabla_{[a_2}H_{b_2]c_2d_2}\nabla_{[a_3}H_{b_3]c_3d_3}R_{a_4b_4c_4d_4}. \tag{5.1}$$

It is notable that, as in [5] where the quartic effective action including B-fields was written by generalising R^4 to $\bar{R}^4$, we can write both the H^2R^3 and H^4R terms as

$$\alpha'^3 \int d^{10}x\sqrt{-g}\, t_8^{a_1b_1\cdots a_4b_4} t_8^{c_1d_1\cdots c_4d_4} H_{a_1c_1e} H_{b_1d_1}{}^{e} \bar{R}_{a_2b_2c_2d_2} \bar{R}_{a_3b_3c_3d_3} \bar{R}_{a_4b_4c_4d_4}, \tag{5.2}$$

where

$$\bar{R}_{abcd} = R_{abcd} + \frac{1}{2} e^{-\phi/2} \nabla_{[a} H_{b]cd}. \tag{5.3}$$

In fact, as in (2.7), [5] also includes an additional term in $\bar{R}_{abcd}$ which involves the dilaton. It is interesting to conjecture that the same may also be true here. This predicts various new terms, such as $H^2R^2(\nabla\nabla\phi)$ and $H^4(\nabla\nabla\phi)^3$. To confirm such terms it would be necessary to calculate five-particle amplitudes involving dilatons, such as the $B^2h^2\phi$ amplitude.

6. The $\epsilon_8\epsilon_8$ terms

So far we have only considered t_8 tensors, where t_8 is formed out of products of delta symbols [6] and which originate from a trace over four R_0^{ab} tensors. This trace, however, also contains an ϵ_8 tensor,

$$\begin{aligned} \mathrm{Tr}(R_0^{ab}R_0^{cd}R_0^{ef}R_0^{gh}) &= \pm\frac{1}{2}\epsilon_8^{abcdefgh} - \frac{1}{2}\delta^{ac}\delta^{bd}\delta^{eg}\delta^{fh} + \cdots \\ &\equiv \pm\frac{1}{2}\epsilon_8^{abcdefgh} + t_8^{abcdefgh}, \end{aligned} \tag{6.1}$$

with the $\pm$ sign depending on the SO(8) chirality. Although momentum conservation requires all terms involving ϵ_8 to vanish in massless $NS \otimes NS$ four-point amplitudes, this is no longer the case for amplitudes involving five states. This leads to various terms in the effective action such as $\epsilon_{10}\epsilon_{10}R^4$ and $B \wedge t_8R^4$. Since the GS light-cone formalism requires $k^+ = 0$ for all external states, the $B \wedge t_8R^4$ term will not be visible. However, terms with at least two contractions between the epsilon tensors should be visible. The $\epsilon_{10}\epsilon_{10}R^4$ term was studied in [1]; here we study equivalent terms involving B-fields, such as $\epsilon_{10}\epsilon_{10}(\nabla H)^2R^2$ and $\epsilon_{10}\epsilon_{10}H^2R^3$.

All t_8 factors in the B^2h^3 amplitude originate from a trace over four R_0^{ab} and so ϵ_8 terms can be included simply by replacing t_8 by (6.1). Since the identities used to reach (3.6) continue to hold for the ϵ_8 terms [1], the final amplitude expanded to lowest order is given by (3.14) but with every occurrence of t_8 in (3.7) and (3.8) replaced by $\frac{1}{2}\epsilon_8 + t_8$. As a consequence of the greater antisymmetry of ϵ_8, all but the final line of A_{rs} vanish and so the ϵ_8 parts of the amplitude contain no massless poles.

In addition to the t_8t_8 terms studied above, the full amplitude now contains $t_8\epsilon_8$ and $\epsilon_8\epsilon_8$ terms. The $t_8\epsilon_8$ must vanish since they do not respect the parities of either the IIA or IIB theory. For the five-graviton amplitude at lowest order in α' this was demonstrated explicitly in [1]; the equivalent statement for B^2h^3 can be shown in an identical manner. So the full amplitude reduces to $(t_8t_8 \pm \frac{1}{4}\epsilon_8\epsilon_8)$ multiplied by the usual kinematic factors and integrals, with $+/-$ for IIB/IIA respectively. As discussed in [1], this calculation appears to give the wrong sign for the $\epsilon_8\epsilon_8$ terms, as can be seen by comparing with the known $\epsilon_{10}\epsilon_{10}R^4$ term in the effective action. We leave this issue unresolved and instead conjecture that $+/-$ should refer to IIA/IIB.

As with the t_8t_8 terms, before any $\epsilon_{10}\epsilon_{10}H^2R^3$ can be determined it is important to subtract diagrams due to quartic terms in the effective action. For the pure Riemann term it is known that the one-loop $t_8t_8R^4$ should be extended to

$$\alpha'^3 \int d^{10}x\sqrt{-g}\, e^{\frac{1}{2}\phi}\left(t_8t_8 \pm \frac{1}{8}\epsilon_{10}\epsilon_{10}\right)R^4, \tag{6.2}$$

with $+/-$ for IIB/IIA respectively [20, 21]. It is natural to conjecture that the $t_8t_8\bar{R}^4$ term in (4.2) is extended in a similar way to $(t_8t_8 \pm \frac{1}{8}\epsilon_{10}\epsilon_{10})\bar{R}^4$, giving the new terms

$$\pm\frac{1}{8}\alpha'^3 \int d^{10}x\sqrt{-g}\, \epsilon_{10\,mn}{}^{a_1b_1a_2b_2a_3b_3a_4b_4}\epsilon_{10}^{mnc_1d_1c_2d_2c_3d_3c_4d_4}\bar{R}_{a_1b_1c_1d_1}\cdots\bar{R}_{a_4b_4c_4d_4}. \tag{6.3}$$

The following analysis will confirm this for a subset of these terms, including $\epsilon_{10}\epsilon_{10}(\nabla H)^4$ and $\epsilon_{10}\epsilon_{10}R^2(\nabla H)^2$.

The expansion of (6.3) around Minkowski space proceeds exactly as in section 4. The only new ingredient is the expansion of the $\epsilon_{10}\epsilon_{10}$ factor. As shown in [1], with the understanding that $\epsilon_{10}\epsilon_{10}$ is always multiplied by a tensor which is symmetric under the interchange of pairs of adjacent indices, i.e. under $(a_r, b_r) \leftrightarrow (a_s, b_s)$, then to first order in h,

$$\begin{aligned}\epsilon_{10\,mn}{}^{a_1b_1\cdots a_4b_4}\epsilon_{10}{}^{mnc_1d_1\cdots c_4d_4} \to &-2\underline{\epsilon}_8^{a_1b_1\cdots a_4b_4}\underline{\epsilon}_8^{c_1d_1\cdots c_4d_4} \\ &+8(h_i{}^{a_1}\underline{\epsilon}_8^{ib_1\cdots a_4b_4}\underline{\epsilon}_8^{c_1d_1\cdots c_4d_4} + h_i{}^{b_1}\underline{\epsilon}_8^{a_1i\cdots a_4b_4}\underline{\epsilon}_8^{c_1d_1\cdots c_4d_4}).\end{aligned} \tag{6.4}$$

Here $\epsilon_{10}\epsilon_{10}$ are 'curved' epsilon tensors which can be rewritten as a sum of products of the metric g, and $\underline{\epsilon}_8\underline{\epsilon}_8$ are 'flat' epsilon tensors which can be rewritten using flat metrics η.

Due to the extra antisymmetry of the epsilon tensor, the four-vertex from (6.3) vanishes and, as a consequence, the $\epsilon_{10}\epsilon_{10}$-equivalents of diagrams 2(a)-(d) all vanish, leaving only diagram 2(e). This tallies with the lack of massless poles in the $\epsilon_8\epsilon_8$ part of the amplitude. However, since the expansion of $\epsilon_{10}\epsilon_{10}$ is so similar to the expansion of t_8t_8, it is prudent to ignore this fact. Then the subtraction of the field theory diagrams from the amplitude proceeds exactly as for the t_8t_8 terms.

After subtracting all diagrams and again using (4.22), but with t_8 replaced by ϵ_8, it is the same five terms in the $\epsilon_8\epsilon_8$-equivalent of (4.21) which remain, leading to two conclusions. Firstly, since the field theory diagrams which must be subtracted involve the $\epsilon_8\epsilon_8BBhh$ and $\epsilon_8\epsilon_8hhh\phi$ vertices, we confirm the presence of both the $\epsilon_{10}\epsilon_{10}R^2(\nabla H)^2$ and $\epsilon_{10}\epsilon_{10}R^3\phi$ terms of (6.3). Secondly, analogous to (4.23), it is necessary to add the term

$$\begin{aligned}\pm\frac{1}{8}\alpha'^3 \int d^{10}x\sqrt{-g}\, &\epsilon_{10\,mn}{}^{a_1b_1a_2b_2a_3b_3a_4b_4}\epsilon_{10}^{mnc_1d_1c_2d_2c_3d_3c_4d_4} \\ &\times H_{a_1c_1e}H_{b_1d_1}{}^eR_{a_2b_2c_2d_2}R_{a_3b_3c_3d_3}R_{a_4b_4c_4d_4},\end{aligned} \tag{6.5}$$

where, as mentioned above, we switch the signs so that $+/-$ applies to IIB/IIA respectively. Since this has the same structure as the $t_8t_8H^2R^3$ term, the full H^2R^3 term can be written as the combination

$$\alpha'^3 \int d^{10}x\sqrt{-g}\left(t_8t_8 \pm \frac{1}{8}\epsilon_{10}\epsilon_{10}\right)H^2R^3, \tag{6.6}$$

where H^2R^3 is shorthand for the above tensor contractions. This mirrors the usual $t_8t_8R^4$ term, which is similarly generalised to $(t_8t_8 \pm \frac{1}{8}\epsilon_{10}\epsilon_{10})R^4$.

The ϵ_8 terms for the Bh^4 amplitude work in an almost identical manner. The field theory diagrams now contain two new vertices, the $\epsilon_8\epsilon_8 BBBB$ and $\epsilon_8\epsilon_8 hBB\phi$ vertices. From these we can confirm the presence of both the $\epsilon_{10}\epsilon_{10}(\nabla H)^4$ and $\epsilon_{10}\epsilon_{10}R(\nabla H)^2\phi$ terms of (6.3). Further, the $t_8t_8H^2(\nabla H)^2R$ term needs to be supplemented by

$$\pm \frac{1}{8}\alpha'^3 \int d^{10}x\sqrt{-g}\,\epsilon_{10\,mn}{}^{a_1b_1a_2b_2a_3b_3a_4b_4}\epsilon_{10}^{mnc_1d_1c_2d_2c_3d_3c_4d_4} \times H_{a_1c_1e}H_{b_1d_1}{}^{e}\nabla_{[a_2}H_{b_2]c_2d_2}\nabla_{[a_3}H_{b_3]c_3d_3}R_{a_4b_4c_4d_4}, \qquad (6.7)$$

and so, as with H^2R^3, the full $H^2(\nabla H)^2R$ term packages together as $\alpha'^3\int d^{10}x\sqrt{-g}\,(t_8t_8 \pm \frac{1}{8}\epsilon_{10}\epsilon_{10})H^2(\nabla H)^2R$.

As mentioned in section 5, it is again notable that the full H^2R^3 and $H^2(\nabla H)^2R$ terms can both be written succinctly as the single term

$$\alpha'^3 \int d^{10}x\sqrt{-g}\left(t_8^{a_1b_1\cdots a_4b_4}t_8^{c_1d_1\cdots c_4d_4} \pm \frac{1}{8}\epsilon_{10\,mn}{}^{a_1b_1\cdots a_4b_4}\epsilon_{10}^{mnc_1d_1\cdots c_4d_4}\right) \times H_{a_1c_1e}H_{b_1d_1}{}^{e}\bar{R}_{a_2b_2c_2d_2}\bar{R}_{a_3b_3c_3d_3}\bar{R}_{a_4b_4c_4d_4}, \qquad (6.8)$$

with $\bar{R}$ given in (5.3).

Acknowledgments

I am very grateful to Michael Green, Hugh Osborn and Pierre Vanhove for many useful discussions.

References

[1] D.M. Richards, *The one-loop five-graviton amplitude and the effective action*, *JHEP* **10** (2008) 042 [arXiv:0807.2421].

[2] M.B. Green and M. Gutperle, *Effects of D-instantons*, *Nucl. Phys.* **B 498** (1997) 195 [hep-th/9701093].

[3] K. Peeters, P. Vanhove and A. Westerberg, *Chiral splitting and world-sheet gravitinos in higher-derivative string amplitudes*, *Class. and Quant. Grav.* **19** (2002) 2699 [hep-th/0112157].

[4] C. Vafa and E. Witten, *A one loop test of string duality*, *Nucl. Phys.* **B 447** (1995) 261 [hep-th/9505053].

[5] D.J. Gross and J.H. Sloan, *The quartic effective action for the heterotic string*, *Nucl. Phys.* **B 291** (1987) 41.

[6] M.B. Green, J.H. Schwarz and E. Witten, *Superstring theory. Volume 2: loop amplitudes, anomalies and phenomenology*, Cambridge University Press, Cambridge U.K. (1987), pg. 596.

[7] E. D'Hoker and D.H. Phong, *Momentum analyticity and finiteness of the one loop superstring amplitude*, *Phys. Rev. Lett.* **70** (1993) 3692 [hep-th/9302003].

JHEP10(2008)043

[8] E. D'Hoker and D.H. Phong, *Dispersion relations in string theory*, *Theor. Math. Phys.* **98** (1994) 306 [hep-th/9404128].

[9] E. D'Hoker and D.H. Phong, *The box graph in superstring theory*, *Nucl. Phys.* **B 440** (1995) 24 [hep-th/9410152].

[10] M.B. Green and P. Vanhove, *The low energy expansion of the one-loop type-II superstring amplitude*, *Phys. Rev.* **D 61** (2000) 104011 [hep-th/9910056].

[11] D.J. Gross and E. Witten, *Superstring modifications of Einstein's equations*, *Nucl. Phys.* **B 277** (1986) 1.

[12] M.B. Green, J.G. Russo and P. Vanhove, *Low energy expansion of the four-particle genus-one amplitude in type-II superstring theory*, *JHEP* **02** (2008) 020 [arXiv:0801.0322].

[13] M.B. Green and P. Vanhove, *D-instantons, strings and M-theory*, *Phys. Lett.* **B 408** (1997) 122 [hep-th/9704145].

[14] M.B. Green and S. Sethi, *Supersymmetry constraints on type IIB supergravity*, *Phys. Rev.* **D 59** (1999) 046006 [hep-th/9808061].

[15] M.B. Green, M. Gutperle and P. Vanhove, *One loop in eleven dimensions*, *Phys. Lett.* **B 409** (1997) 177 [hep-th/9706175].

[16] M.B. Green, J.H. Schwarz and E. Witten, *Superstring theory. Volume* 1*: introduction*, Cambridge University Press, Cambridge U.K. (1987), pg. 469.

[17] S. Sannan, *Gravity as the limit of the type II superstring theory*, *Phys. Rev.* **D 34** (1986) 1749.

[18] A. Kehagias and H. Partouche, *On the exact quartic effective action for the type IIB superstring*, *Phys. Lett.* **B 422** (1998) 109 [hep-th/9710023].

[19] S. Frolov, I.R. Klebanov and A.A. Tseytlin, *String corrections to the holographic RG flow of supersymmetric* $\mathrm{SU}(N) \times \mathrm{SU}(N+M)$ *gauge theory*, *Nucl. Phys.* **B 620** (2002) 84 [hep-th/0108106].

[20] E. Kiritsis and B. Pioline, *On* R^4 *threshold corrections in type IIB string theory and* (p, q) *string instantons*, *Nucl. Phys.* **B 508** (1997) 509 [hep-th/9707018].

[21] I. Antoniadis, S. Ferrara, R. Minasian and K.S. Narain, R^4 *couplings in M- and type-II theories on Calabi-Yau spaces*, *Nucl. Phys.* **B 507** (1997) 571 [hep-th/9707013].

JHEP10(2008)043